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Fractional variational calculus in terms of Riesz fractional derivatives

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Abstract

This paper presents extensions of traditional calculus of variations for systems containing Riesz fractional derivatives (RFDs). Specifically, we present generalized Euler–Lagrange equations and the transversality conditions for fractional variational problems (FVPs) defined in terms of RFDs. We consider two problems, a simple FVP and an FVP of Lagrange. Results of the first problem are extended to problems containing multiple fractional derivatives, functions and parameters, and to unspecified boundary conditions. For the second problem, we present Lagrange-type multiplier rules. For both problems, we develop the Euler–Lagrange-type necessary conditions which must be satisfied for the given functional to be extremum. Problems are considered to demonstrate applications of the formulations. Explicitly, we introduce fractional momenta, fractional Hamiltonian, fractional Hamilton equations of motion, fractional field theory and fractional optimal control. The formulations presented and the resulting equations are similar to the formulations for FVPs given in Agrawal (2002 *J. Math. Anal. Appl.* **272** 368, 2006 *J. Phys. A: Math. Gen.* **39** 10375) and to those that appear in the field of classical calculus of variations. These formulations are simple and can be extended to other problems in the field of fractional calculus of variations.

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1. Introduction

This paper deals with fractional derivatives in general and fractional calculus of variations (FCV) in particular. Fractional derivatives (or fractional calculus) have recently played a very important role in various fields such as mechanics, electricity, chemistry, biology, economics, control theory, robotics, image and signal processing, and anomalous transport and diffusion

[3–10]. This list is by no means complete, and many more applications could be found in above references and the citations therein.

In recent years, there has been a growing interest in the area of FCV. Riewe's two papers [11, 12] are the starting point of this field. In his papers, Riewe argued that the second-order derivative terms in Euler–Lagrange equations come from quadratic first derivative terms in Lagrangians, and therefore the first-order derivative terms should come from quadratic half-order derivative terms in Lagrangians. Subsequently, he used fractional calculus to develope Lagrangian, Hamiltonian and other concepts of classical mechanics for nonconservative systems. Agrawal [13] presented a heuristic approach to obtain differential equations of fractionally damped systems. Later, Agrawal [1] presented generalized Euler–Lagrange equations (GELEs) for unconstrained and constrained fractional variational problems (FVPs). Klimek presented a fractional sequential mechanics model with symmetric fractional derivatives [14] and stationary conservation laws for fractional differential equations with variable coefficients [15]. Dreisigmeyer and Young [16] presented nonconservative Lagrangian mechanics using a generalized function approach. In [17], the authors show that obtaining differential equations for a nonconservative system using fractional variational calculus may not be possible. Cresson [18] presented fractional embedding of differential operators and Lagrangian systems where he shows that the embedding procedure is compatible with FCV. He also develops several formulations for fractional mechanics.

The papers cited above focus on obtaining Euler–Lagrange equations for FVPs, but they do not consider initial or terminal conditions. In [2, 19], Agrawal argued that integer variational calculus provides not only the differential equations for the problems, but it also provides the transversality conditions which give the natural boundary conditions (NBCs). Therefore, FCV should also do the same. Subsequently, he developed the GELEs and the transversality conditions for FVPs. In these papers, the problems were formulated in terms of the Riemann–Liouville and the Caputo fractional derivatives. It was shown that both derivatives may appear in the resulting GELEs even when a problem is formulated in terms of only one of them.

The fractional Euler–Lagrange equation has recently been used by Baleanu and coworker to model fractional Lagrangian and Hamiltonian formulations with linear velocities [20, 21], Hamiltonian equations for FVPs [22–24] and fractional Hamiltonian analysis of higher order derivative systems [23]. Agrawal [26–28] and Agrawal and Baleanu [29] presented formulations for deterministic and stochastic analyses of fractional optimal control problems. Tarasov and Zaslavsky [30] have used variational Euler–Lagrange equations to derive fractional generalization of the Ginzburg–Landau equation for fractal media. In [31], the authors use fractional variational principles to develop fractional generalization of nonholonomic constraints. Stanislavsky [32] presented analysis of a simple fractional and a coupled fractional oscillators, and a generalization of classical mechanics with fractional derivatives. In [33], a variational principle for a more general set of equations is formulated and the existence and uniqueness of a second-order differential equation containing the fractional order viscoelastic model has been provided.

The above list clearly suggests that interest in the fractional variational calculus is growing; however, much remains to be done. For example, FCV began with a desire to find a Lagrangian that could provide viscous damping force. In spite of several efforts, a satisfactory Lagrangian that could accomplish the above has not yet been (to the author's knowledge) presented. Our conjecture is that answers to this and other unsolved problems in fractional mechanics, or for that matter in many other fields, lie in an FCV developed using fractional derivatives which may or may not have been defined at the present time. With this in mind, our effort to develop FCV and its applications in various fields using various fractional derivatives must continue.

In this paper, we develop the GELEs and the transversality conditions for FVPs defined in terms of Riesz fractional derivatives (RFDs). Thus, it extends the FCV available to researchers so far. Definition of a Riesz fractional potential is used to define an RFD. Two definitions are possible for an RFD, one analogous to the Riemann–Liouville fractional derivative (RLFD) and the other analogous to the Caputo fractional derivative (CFD). Both definitions are presented. We consider two problems, a simple FVP and an FVP of Lagrange. Results of the first problem are extended to problems containing multiple fractional derivatives, functions and parameters, and to unspecified boundary conditions. For the second problem, we present a Lagrange-type multiplier rules. Problems are considered to demonstrate applications of the formulations. We introduce fractional momenta, fractional Hamiltonian, fractional Hamilton equations of motion, fractional field theory and fractional optimal control. The formulations presented and the resulting equations are similar to the formulations for FVPs given in [1, 2] and to those that appear in the field of classical calculus of variations. These formulations are simple and can be extended to other problems in the field of FCV.

We begin with some definitions of fractional integrals and derivatives, and their basic properties.

2. Fractional integrals and derivatives and their properties

Several definitions of a fractional derivative have been proposed. These definitions include the Riemann–Liouville, the Grunwald–Letnikov, the Weyl, the Caputo, the Marchaud, the Riesz, and the Miller and Ross fractional derivatives (see [1] and the references therein). In this section, we present definitions of the Riesz fractional integral (potential) and derivatives, and their properties. We also define the Riemann–Liouville and Caputo derivatives as they are linked to Riesz fractional derivatives.

We begin with the left and the right Riemann–Liouville fractional integrals of order $\alpha > 0$ of a function $x(t)$ which are defined as [4]

$$ {}_aI_t^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau)\, \mathrm{d}\tau, \qquad \alpha > 0 \tag{1} $$

and

$$ {}_tI_b^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} x(\tau)\, \mathrm{d}\tau, \qquad \alpha > 0, \tag{2} $$

where $\Gamma(*)$ represents the Gamma function. We now define the Riesz potential as

$$ {}_a^RI_b^\alpha x(t) = \frac{1}{2\Gamma(\alpha)} \int_a^b |t-\tau|^{\alpha-1} x(\tau)\, \mathrm{d}\tau, \qquad \alpha > 0. \tag{3} $$

Note that the definitions of ${}_a^RI_b^\alpha x(t)$ in [34] include an additional factor of $1/\cos(\pi\alpha/2)$ and in [4] a factor of 2. The rationale for considering the above definition will be given shortly. Strictly speaking, the limits in Riesz potential go from $-\infty$ to $+\infty$. Therefore, equation (3) is valid for only a special class of functions or it can be considered as a Riesz-type potential. Nevertheless, we call it Riesz potential and also the Riesz fractional integral of order α. From equations (1)–(3), it follows that

$$ {}_a^RI_b^\alpha x(t) = \tfrac{1}{2}\left({}_aI_t^\alpha x(t) + {}_tI_b^\alpha x(t)\right). \tag{4} $$

Operator ${}_a^RI_b^\alpha$ satisfies the following identity:

$$ Q{}_a^RI_b^\alpha x(t) = {}_a^RI_b^\alpha Qx(t), \tag{5} $$

where Q is the '*reflection operator*' such that $(Qx)(t) = x(a+b-t)$. This follows from equation (4) and the identities $Q_a I_t^\alpha = {}_t I_b^\alpha Q$ and $Q_t I_b^\alpha = {}_a I_t^\alpha Q$ given in [34]. Thus, the operators Q and ${}_a^R I_b^\alpha$ commute.

Using equations (1) and (2), the left and the right Riemann–Liouville and Caputo derivatives are defined as the left Riemann–Liouville fractional derivative (RLFD)

$$ {}_a D_t^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} x(\tau)\,\mathrm{d}\tau = D^n {}_a I_t^{n-\alpha} x(t), \tag{6} $$

the right Riemann–Liouville fractional derivative (RLFD)

$$ {}_t D_b^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_t^b (\tau-t)^{n-\alpha-1} x(\tau)\,\mathrm{d}\tau = (-D)^n {}_t I_b^{n-\alpha} x(t), \tag{7} $$

the left Caputo fractional derivative (CFD)

$$ {}_a^C D_t^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n x(\tau)\,\mathrm{d}\tau = {}_a I_t^{n-\alpha} D^n x(t), \tag{8} $$

the right Caputo fractional derivative (CFD)

$$ {}_t^C D_b^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1} \left(-\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n x(\tau)\,\mathrm{d}\tau = {}_t I_b^{n-\alpha} (-D)^n x(t), \tag{9} $$

where D is the traditional derivative operator and α is the order of the derivative such that $n-1 < \alpha < n$. When α is an integer, the usual definition of a derivative is used. Note that for $\alpha = 1$, the left derivative is the negative of the right derivative.

From equations (6) and (9), it is clear that both the Riemann–Liouville and the Caputo derivatives contain fractional integration of order $n-\alpha$ and traditional derivative of order n. However, in the case of Riemann–Liouville, the integration is performed first and the differentiation is performed next, whereas in the case of Caputo derivative, the order of differentiation and integration is reversed. Following the above analogy, we define the fractional Riesz and fractional Riesz–Caputo derivatives as [4] the Riesz fractional derivative (RFD)

$$ {}_a^R D_t^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^b |t-\tau|^{n-\alpha-1} x(\tau)\,\mathrm{d}\tau = D^n {}_a^R I_t^{n-\alpha} x(t), \tag{10} $$

the Riesz–Caputo fractional derivative (RCFD)

$$ {}_a^{\mathrm{RC}} D_t^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^b |t-\tau|^{n-\alpha-1} \left(\frac{\mathrm{d}}{\mathrm{d}\tau}\right)^n x(\tau)\,\mathrm{d}\tau = {}_a^R I_t^{n-\alpha} D^n x(t). \tag{11} $$

Equation (11) essentially represents the Riesz Riemann–Liouville fractional derivative. However, in the discussion below, we simply call it the Riesz fractional derivative.

Using equations (4) and (6)–(11), it follows that

$$ {}_a^R D_t^\alpha x(t) = \tfrac{1}{2}\left({}_a D_t^\alpha x(t) + (-1)^n {}_t D_b^\alpha x(t)\right) \tag{12} $$

and

$$ {}_a^{\mathrm{RC}} D_t^\alpha x(t) = \tfrac{1}{2}\left({}_a^C D_t^\alpha x(t) + (-1)^n {}_t^C D_b^\alpha x(t)\right). \tag{13} $$

In particular, for $0 < \alpha < 1$, we have

$$ {}_a^R D_t^\alpha x(t) = \tfrac{1}{2}\left({}_a D_t^\alpha x(t) - {}_t D_b^\alpha x(t)\right) \tag{14} $$

and

$$ {}_a^{\mathrm{RC}} D_t^\alpha x(t) = \tfrac{1}{2}\left({}_a^C D_t^\alpha x(t) - {}_t^C D_b^\alpha x(t)\right). \tag{15} $$

As pointed out earlier, when α is 1, the right derivative is the negative of the left derivative. Thus, for integer α, the Riesz derivatives defined above agree with traditional definitions of a derivative. This is the rationale for using equation (3) as the definition for the Riesz fractional integral.

In the discussion to follow, we will also need formulae for fractional integration by parts. These formulae are given as

$$\int_a^b [\star]\, {}_a^C D_t^\alpha \eta \, \mathrm{d}t = \int_a^b \eta \, {}_t D_b^\alpha [\star] \, \mathrm{d}t + \sum_{j=0}^{n-1} {}_t D_b^{\alpha+j-n} [\star] D^{n-1-j} \eta(t)|_a^b, \tag{16}$$

$$\int_a^b [\star]\, {}_t^C D_b^\alpha \eta \, \mathrm{d}t = \int_a^b \eta \, {}_a D_t^\alpha [\star] \, \mathrm{d}t + \sum_{j=0}^{n-1} (-1)^{n+j} {}_a D_t^{\alpha+j-n} [\star] D^{n-1-j} \eta(t)|_a^b, \tag{17}$$

$$\int_a^b [\star]\, {}_a D_t^\alpha \eta \, \mathrm{d}t = \int_a^b \eta \, {}_t^C D_b^\alpha [\star] \, \mathrm{d}t - \sum_{j=0}^{n-1} (-1)^{n+j} {}_a D_t^{\alpha+j-n} \eta(t) D^{n-1-j} [\star]|_a^b, \tag{18}$$

$$\int_a^b [\star]\, {}_t D_b^\alpha \eta \, \mathrm{d}t = \int_a^b \eta \, {}_a^C D_t^\alpha [\star] \, \mathrm{d}t - \sum_{j=0}^{n-1} {}_t D_b^{\alpha+j-n} [\star] D^{n-1-j} \eta(t)|_a^b, \tag{19}$$

$$\begin{aligned} \int_a^b [\star]\, {}_a^{\mathrm{RC}} D_b^\alpha \eta \, \mathrm{d}t &= (-1)^n \int_a^b \eta \, {}_a^R D_b^\alpha [\star] \, \mathrm{d}t \\ &\quad + \sum_{j=0}^{n-1} (-1)^j \, {}_a^R D_b^{\alpha+j-n} [\star] D^{n-1-j} \eta(t)|_a^b, \end{aligned} \tag{20}$$

$$\begin{aligned} \int_a^b [\star]\, {}_a^R D_b^\alpha \eta \, \mathrm{d}t &= (-1)^n \int_a^b \eta \, {}_a^{\mathrm{RC}} D_b^\alpha [\star] \, \mathrm{d}t \\ &\quad + \sum_{j=0}^{n-1} (-1)^{n+j} \, {}_a^R D_b^{\alpha+j-n} \eta(t) D^{n-1-j} [\star]|_a^b. \end{aligned} \tag{21}$$

Equations (16) and (17) could be found in [19]. On the other hand, they could also be derived using equations (7) and (9), the identity (see [34])

$$\int_a^b f \, {}_a I_t^\alpha g \, \mathrm{d}t = \int_a^b g \, {}_t I_b^\alpha f \, \mathrm{d}t,$$

and performing integration by parts. Equations (18) and (19) follow from equations (16) and (17) by interchanging $[*]$ and η, and rearranging the terms. Finally, equations (20) and (21) follow from equations (12), (13) and (16)–(19). Note that a negative exponent in ${}_a^R D_b^\alpha$ and ${}_a^{\mathrm{RC}} D_b^\alpha$ represents a Riesz fractional integral.

The above equations play an important role in deriving the GELEs for fractional variational problems defined in terms of RFDs and RCFDs. The procedures to derive formulations and the resulting equations for problems defined in terms of RFDs and RCFDs turn out to be very similar. For this reason, we give derivations and proofs for problems defined in terms of RCFDs only, and we give only the theorems and the results without any derivations or proof for problems defined in terms of RFDs.

We now derive the GELEs for a simple fractional variational problem defined in terms of an RCFD.

3. Euler–Lagrange equation for a simple fractional variational problem

A simple FVP in terms of an RCFD can be defined as follows: *among all functions $x(t)$ which satisfy the boundary conditions*

$$x(a) = x_a \qquad \text{and} \qquad x(b) = x_b \tag{22}$$

find the function for which the functional

$$J[x] = \int_a^b F\left(t, x, {}_a^{\mathrm{RC}}D_b^\alpha x\right) \mathrm{d}t \tag{23}$$

is an extremum, where $0 < \alpha < 1$. We also assume that functions are smooth and all differentiability conditions are met.

To obtain the necessary conditions for the extremum, assume that $x^*(t)$ is the desired function. Let $\epsilon \in R$, and define a family of curves

$$x(t) = x^*(t) + \epsilon\eta(t), \tag{24}$$

where $\eta(t)$ is an arbitrary curve except that it satisfies the boundary conditions, i.e. we require that

$$\eta(a) = \eta(b) = 0. \tag{25}$$

To obtain the Euler–Lagrange equation, we substitute equation (24) into equation (23), differentiate the resulting equation with respect to ϵ and set the result to 0. This leads to the following condition for extremum:

$$\int_a^b \left[\frac{\partial F}{\partial x}\eta + \frac{\partial F}{\partial {}_a^{\mathrm{RC}}D_b^\alpha x}\, {}_a^{\mathrm{RC}}D_b^\alpha \eta\right] \mathrm{d}t = 0. \tag{26}$$

Using equation (20), equation (26) can be written as

$$\int_a^b \left[\frac{\partial F}{\partial x} - {}_a^R D_b^\alpha \frac{\partial F}{\partial {}_a^{\mathrm{RC}}D_b^\alpha x}\right] \eta\, \mathrm{d}t + {}_a^R D_b^{\alpha-1}\left(\frac{\partial F}{\partial {}_a^{\mathrm{RC}}D_b^\alpha x}\right)\eta(t)\Big|_a^b = 0. \tag{27}$$

Using equation (25), it follows that the last two terms of equation (27) are 0. Since $\eta(t)$ is arbitrary, it follows from a well established result in calculus of variations that [35]

$$\frac{\partial F}{\partial x} - {}_a^R D_b^\alpha \frac{\partial F}{\partial {}_a^{\mathrm{RC}}D_b^\alpha x} = 0. \tag{28}$$

Equation (28) is the *generalized Euler–Lagrange equation* for the FCV problem defined in terms of the RCFD. Note that both the RFD and the RCFD automatically appear in the resulting differential equations even when the functional contains only the RCFD.

Following the above derivation, we can state the following theorem:

Theorem 1. *Let $J[x]$ be a functional of the form*

$$\int_a^b F\left(t, x, {}_a^{\mathrm{RC}}D_b^\alpha x\right) \mathrm{d}t, \tag{29}$$

defined on the set of functions $x(t)$ which have continuous Riesz–Caputo fractional derivative of order α in $[a, b]$ and which satisfy the boundary conditions $x(a) = x_a$ and $x(b) = x_b$. Then a necessary condition for $J[x]$ to have an extremum for a given function $x(t)$ is that $x(t)$ satisfy the following generalized Euler–Lagrange equation:

$$\frac{\partial F}{\partial x} - {}_a^R D_b^\alpha \frac{\partial F}{\partial {}_a^{\mathrm{RC}}D_b^\alpha x} = 0, \tag{30}$$

Function F in equation (23) can be thought of as a function containing both the left and the right CFDs for which the GELE is given as [19]

$$\frac{\partial F}{\partial x} + {}_tD_b^\alpha \frac{\partial F}{\partial {}_a^CD_t^\alpha x} + {}_aD_t^\alpha \frac{\partial F}{\partial {}_t^CD_b^\alpha x} = 0. \tag{31}$$

Equation (30) can also be obtained by using equations (14), (15) and (31). For $\alpha = 1$, the Euler–Lagrange equation is given as

$$\frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial \dot{x}} = 0. \tag{32}$$

Equations (30), (31) (and its equivalent in terms of RLFD) and (32) are similar and they all contain both forward and backward derivatives. Note that $-\mathrm{d}/\mathrm{d}t$ is essentially a backward derivative. Thus, backward derivatives in equations (30) and (31) appear explicitly, whereas they appear in equation (32) in a disguise form.

For $F = F\big(t, x, {}_a^RD_b^\alpha x\big)$ and $F = F\big(t, x, {}_a^{\mathrm{RC}}D_b^\alpha x, {}_a^RD_b^\alpha x\big)$, the corresponding Euler–Lagrange equations are given as

$$\frac{\partial F}{\partial x} - {}_a^{\mathrm{RC}}D_b^\alpha \frac{\partial F}{\partial {}_a^RD_b^\alpha x} = 0 \tag{33}$$

and

$$\frac{\partial F}{\partial x} - {}_a^RD_b^\alpha \frac{\partial F}{\partial {}_a^{\mathrm{RC}}D_b^\alpha x} - {}_a^{\mathrm{RC}}D_b^\alpha \frac{\partial F}{\partial {}_a^RD_b^\alpha x} = 0, \tag{34}$$

respectively. The derivations of these equations are omitted as they are similar to those of equation (28).

In the next section, we consider the case where a boundary condition may not be specified.

4. A simple variable end-point problem

We now consider a simple variable end-point problem in FCV which can be defined as follows: *among all possible curves whose end points lie on two given vertical lines $t = a$ and $t = b$, find the curve for which the functional defined in equation* (23) *has an extremum.*

To obtain the necessary conditions for this problem, we follow the approach discussed in section 2 and arrive at equation (27). Since $\eta(t)$ is arbitrary, we first select $\eta(t)$ such that $\eta(a) = \eta(b) = 0$. Using this condition and equation (27), we arrive at equation (28). Substituting equation (28) into equation (27), we obtain

$${}_a^RD_b^{\alpha-1}\left(\frac{\partial F}{\partial {}_a^{\mathrm{RC}}D_b^\alpha x}\right)\eta(t)\big|_a^b = 0. \tag{35}$$

Equation (35) suggests that at $t = a$ either $\eta(a)$ should be zero (i.e. $x(a)$ should be specified) or the following condition must be met:

$${}_a^RD_b^{\alpha-1}\left(\frac{\partial F}{\partial {}_a^{\mathrm{RC}}D_b^\alpha x}\right)\bigg|_{t=a} = 0. \tag{36}$$

A similar condition holds at $t = b$. Conditions $x(a) = x_a$ and $x(b) = x_b$ are known as the *geometric boundary conditions* (GBC), and equation (36) and its equivalent for point $t = b$ are called the *generalized natural boundary conditions* (GNBCs). Thus, the solution of the variable end-point problem must satisfy the GELE (28) and either the GBCs or the GNBCs at both ends. Equation (35) also suggests that both the GBCs and the GNBCs cannot be specified at the same end point, otherwise, the problem will be inconsistent. Equation (35) may contain fractional derivative terms. Therefore, solution of a fractional differential equation (FDE) resulting from FCV may require fractional boundary conditions.

Following the above approach, it can be shown that if $F = F\big(t, x, {}_a^R D_b^\alpha x\big)$, equation (35) is replaced with

$$\left(\frac{\partial F}{\partial {}_a^R D_b^\alpha x}\right) {}_a^R D_b^{\alpha-1} \eta(t)|_a^b = 0. \tag{37}$$

Thus, in this case both the GBC and the GNBC may contain fractional derivative terms. If $F = F\big(t, x, {}_a^{\mathrm{RC}} D_b^\alpha x, {}_a^R D_b^\alpha x\big)$, then equation (35) is replaced with

$$\left[{}_a^R D_b^{\alpha-1}\left(\frac{\partial F}{\partial {}_a^{\mathrm{RC}} D_b^\alpha x}\right)\eta(t) - \left(\frac{\partial F}{\partial {}_a^R D_b^\alpha x}\right){}_a^R D_b^{\alpha-1}\eta(t)\right]\Bigg|_a^b = 0. \tag{38}$$

In this case, the GBCs and the GNBCs are not simple. For this reason, we will not consider a function F which contains both derivatives, ${}_a^{\mathrm{RC}} D_b^\alpha x$ and ${}_a^R D_b^\alpha x$. However, if necessary, the formulations presented here can be extended to the case where $F = F\big(t, x, {}_a^{\mathrm{RC}} D_b^\alpha x, {}_a^R D_b^\alpha x\big)$.

The above formulations consider only one RCFD. The problem of finding extremum of a functional consisting of multiple Riesz fractional derivatives of order less than or equal to 1 and mixed boundary conditions can be developed using the discussion presented in this section and the previous one. This leads to

Theorem 2. *Assume that* $0 < \alpha_1, \ldots, \alpha_m < 1$. *Let* $J[x]$ *be a functional of the form*

$$\int_a^b F\big(t, x, {}_a^{\mathrm{RC}} D_b^{\alpha_1} x, \ldots, {}_a^{\mathrm{RC}} D_b^{\alpha_m} x\big)\,\mathrm{d}t, \tag{39}$$

defined on the set of functions $x(t)$ *which have continuous first and second partial derivatives with respect to all its arguments in* $[a, b]$. *Then a necessary condition for* $J[x]$ *to have an extremum* $x(t)$ *is that* $x(t)$ *satisfy the generalized Euler–Lagrange equation*

$$\frac{\partial F}{\partial x} - \sum_{i=1}^m {}_a^R D_b^{\alpha_i} \frac{\partial F}{\partial {}_a^{\mathrm{RC}} D_b^{\alpha_i} x} = 0, \tag{40}$$

either $x(a) = x_a$ *or*

$$\sum_{i=1}^m {}_a^R D_b^{\alpha_i-1}\left(\frac{\partial F}{\partial {}_a^{\mathrm{RC}} D_b^{\alpha_i} x}\right)\Bigg|_{t=a} = 0, \tag{41}$$

and either $x(b) = x_b$ *or*

$$\sum_{i=1}^m {}_a^R D_b^{\alpha_i-1}\left(\frac{\partial F}{\partial {}_a^{\mathrm{RC}} D_b^{\alpha_i} x}\right)\Bigg|_{t=b} = 0. \tag{42}$$

Equations (40)–(42) can be derived using the approach discussed in this and in the previous section. In the case of $F = F\big(t, x, {}_a^R D_b^{\alpha_1} x, \ldots, {}_a^R D_b^{\alpha_m} x\big)$, equations (40)–(42) take the following form:

$$\frac{\partial F}{\partial x} - \sum_{i=1}^m {}_a^{\mathrm{RC}} D_b^{\alpha_i} \frac{\partial F}{\partial {}_a^R D_b^{\alpha_i} x} = 0, \tag{43}$$

$$\sum_{i=1}^m {}_a^R D_b^{\alpha_i-1}\eta(t)\left(\frac{\partial F}{\partial {}_a^{\mathrm{RC}} D_b^{\alpha_i} x}\right)\Bigg|_{t=a} = 0 \tag{44}$$

and

$$\sum_{i=1}^{m} {}_a^R D_b^{\alpha_i - 1} \eta(t) \left(\frac{\partial F}{\partial {}_a^{\mathrm{RC}} D_b^{\alpha_i} x} \right) \bigg|_{t=b} = 0. \tag{45}$$

Here $\eta(t)$ is an arbitrary function consistent with boundary conditions. Note that in this case separating the GBCs and the GNBCs is not clear.

We now give further generalization of theorems 1 and 2.

5. The cases of multiple functions and multiple α's greater than 1

The forgoing derivations can be generalized for the cases of multiple functions and multiple α_j, $j = 1, \ldots,$ where at least one of the α's is greater than 1. To simplify the presentation, we consider the following two cases in which the first case has n functions, $n > 1$, and only one α, $0 < \alpha < 1$, and the second case has only one function and one α greater than 1. In both cases, we assume that the locations of the end points are fixed, and some functions and their fractional derivatives are either specified or free (unknown). A formulation in which the functional contains multiple functions and multiple derivatives of arbitrary order and mixed boundary conditions can be developed following the approach presented here. A more general case, where the end points are free would be presented elsewhere.

Case 1. Multiple functions, one α, $0 < \alpha < 1$, *and mixed boundary conditions*

The fractional variational problems discussed in the previous sections can be extended in a straightforward manner to problems where the functionals have more than one RCFD or more than one RFD. We first consider the problem in which F contains $x_j(t)$, $j = 1, \ldots, n$, and its RCFDs only. To define a problem for this case, consider an integer $n > 1$, and a set S_n consisting of integers from 1 to n. Consider four additional subsets S_L, S_R, $\bar{S}_L$ and $\bar{S}_R$ where S_L and S_R consist of some of the integer numbers between 0 and $n + 1$, $\bar{S}_L = S_n - S_L$ and $\bar{S}_R = S_n - S_R$. Some of these sets could be empty. For example, assume that we have three functions, $x_1(t)$, $x_2(t)$ and $x_3(t)$. Assume that x_2 is defined at a and all three x's are defined at b. In this case, the above sets are given as $S_n = \{1, 2, 3\}$, $S_L = \{2\}$, $S_R = \{1, 2, 3\}$, $\bar{S}_L = \{1, 3\}$ and $\bar{S}_R = \emptyset$, where $\emptyset$ is the empty set. The problem can now be defined as follows: let $F(t, x_1, \ldots, x_n, z_1, \ldots, z_n)$ be a function with continuous first and second (partial) derivatives with respect to all its arguments. For $0 < \alpha < 1$, consider the problem of finding necessary conditions for an extremum of a functional of the form

$$\int_a^b F\big(t, x_1, \ldots, x_n, {}_a^{\mathrm{RC}} D_b^{\alpha} x_1, \ldots, {}_a^{\mathrm{RC}} D_b^{\alpha} x_n\big)\, \mathrm{d}t, \tag{46}$$

which depends on n continuously differentiable functions $x_1(t), \ldots, x_n(t)$ satisfying the boundary conditions

$$x_j(a) = x_{ja}, \qquad j \in S_L, \tag{47}$$

$$x_j(b) = x_{jb}, \qquad j \in S_R. \tag{48}$$

Note that all functions $x_j(t)$, $j = 1, \ldots, n$, are independent. Therefore, the necessary condition for the functional in equation (46) to have an extremum can be found by considering the variations of each function one at a time. Thus, we have

Theorem 3. *A necessary condition for the curve* $x_j = x_j(t)$, $j = 1, \ldots, n$, *which satisfies the boundary conditions given by equations (47) and (48) to be extremal of the functional given by* *equation (46) is that the functions $x_j = x_j(t)$, $j = 1, \dots, n$, satisfy the following generalized Euler–Lagrange equations:*

$$\frac{\partial F}{\partial x_j} - {}_a^R D_b^\alpha \frac{\partial F}{\partial {}_a^{\mathrm{RC}} D_b^\alpha x_j} = 0, \qquad j = 1, \dots, n, \tag{49}$$

and the following generalized natural boundary conditions:

$${}_a^R D_b^{\alpha-1} \left(\frac{\partial F}{\partial {}_a^{\mathrm{RC}} D_b^\alpha x_j} \right) \bigg|_{t=a} = 0, \qquad j \in \bar{S}_L \tag{50}$$

$${}_a^R D_b^{\alpha-1} \left(\frac{\partial F}{\partial {}_a^{\mathrm{RC}} D_b^\alpha x_j} \right) \bigg|_{t=b} = 0, \qquad j \in \bar{S}_R. \tag{51}$$

If F in equation (46) is given as $F = F\big(t, x_1, \dots, x_n, {}_a^R D_b^\alpha x_1, \dots, {}_a^R D_b^\alpha x_n\big)$, then the necessary conditions equivalent to equations (49)–(51) are given as

$$\frac{\partial F}{\partial x_j} - {}_a^{\mathrm{RC}} D_b^\alpha \frac{\partial F}{\partial {}_a^R D_b^\alpha x_j} = 0, \qquad j = 1, \dots, n, \tag{52}$$

$${}_a^R D_b^{\alpha-1} \eta_j(t) \left(\frac{\partial F}{\partial {}_a^R D_b^\alpha x_j} \right) \bigg|_{t=a} = 0, \qquad j \in \bar{S}_L \tag{53}$$

$${}_a^R D_b^{\alpha-1} \eta_j(t) \left(\frac{\partial F}{\partial {}_a^R D_b^\alpha x_j} \right) \bigg|_{t=b} = 0, \qquad j \in \bar{S}_R. \tag{54}$$

If $F = F\big(t, x_1, \dots, x_n, {}_a^{\mathrm{RC}} D_b^\alpha x_1, \dots, {}_a^{\mathrm{RC}} D_b^\alpha x_n\big)$, then the GBCs do not contain fractional derivative terms, and if $F = F\big(t, x_1, \dots, x_n, {}_a^R D_b^\alpha x_1, \dots, {}_a^R D_b^\alpha x_n\big)$, then both the GBCs and the GNBCs may contain fractional derivative terms.

We now consider the case of higher order derivatives.

Case 2. One function and its derivatives of order $\alpha > 1$, and mixed boundary conditions

Now consider the case where the functional contains an RCFD of order $\alpha > 1$. Let n be an integer such that $n - 1 < \alpha < n$. This time, define sets S_n, S_L, S_R, $\bar{S}_L$ and $\bar{S}_R$ as follows: $S_n = \{0, \dots, n-1\}$. If $x^{(j)}(a)$ is specified then $j \in S_L$, otherwise $j \in \bar{S}_L$. Similarly, if $x^{(j)}(b)$ is specified then $j \in S_R$, otherwise $j \in \bar{S}_R$. We further assume that $F(t, x, z)$ is a function with continuous first and second (partial) derivatives with respect to all its arguments, and consider a functional of the form

$$J[x] = \int_a^b F\big(t, x, {}_a^{\mathrm{RC}} D_b^\alpha x\big)\, \mathrm{d}t. \tag{55}$$

The problem can now be defined as follows: *among all functions $x(t)$ satisfying the conditions*

$$x^{(j)}(a) = x_{aj}, \qquad j \in S_L \tag{56}$$

$$x^{(j)}(b) = x_{bj}, \qquad j \in S_R, \tag{57}$$

find the function for which functional $J[x]$ defined by (55) has an extremum.

To find the necessary conditions for this problem, we define $x(t)$ as in equation (24), substitute it into equation (55), differentiate the resulting equations with respect to ϵ and set it to 0, and use equation (20) to eliminate terms containing fractional derivatives of $\eta(t)$. We further impose the condition that $\eta(t)$ is arbitrary, and $D^j \eta(a)$ $(D^j \eta(b))$ is arbitrary if

$D^j x(a)$ $(D^j x(b))$ is not specified, and therefore their coefficients must be 0. This leads to the following theorem:

Theorem 4. *Let $J[x]$ be a functional of the form given by equation (55) defined on the set of functions satisfying the boundary conditions given by equations (56) and (57). Then a necessary condition for $J[x]$ to have an extremum for a given function $x(t)$ is that $x(t)$ satisfy the generalized Euler–Lagrange equation*

$$\frac{\partial F}{\partial x} + (-1)^n\, {}^{R}_{a}D^{\alpha}_{b} \frac{\partial F}{\partial\, {}^{\mathrm{RC}}_{a}D^{\alpha}_{b} x} = 0 \tag{58}$$

and the generalized natural boundary conditions

$$ {}^{R}_{a}D^{\alpha+j-n}_{b} \left(\frac{\partial F}{\partial\, {}^{\mathrm{RC}}_{a}D^{\alpha}_{b} x} \right)\Bigg|_{t=a} = 0, \qquad (n-1-j) \in \bar{S}_L \tag{59}$$

$$ {}^{R}_{a}D^{\alpha+j-n}_{b} \left(\frac{\partial F}{\partial\, {}^{\mathrm{RC}}_{a}D^{\alpha}_{b} x} \right)\Bigg|_{t=b} = 0, \qquad (n-1-j) \in \bar{S}_R. \tag{60}$$

In the case of $F = F\big(t, x, {}^{R}_{a}D^{\alpha}_{b} x\big)$, $0 < n-1 < \alpha < n$, the necessary conditions for extremum are given as

$$\frac{\partial F}{\partial x} + (-1)^n\, {}^{\mathrm{RC}}_{a}D^{\alpha}_{b} \frac{\partial F}{\partial\, {}^{R}_{a}D^{\alpha}_{b} x} = 0. \tag{61}$$

$$\left(\frac{\partial F}{\partial\, {}^{\mathrm{RC}}_{a}D^{\alpha}_{b} x} \right)\Bigg|_{t=a} = 0, \qquad (n-1-j) \in \bar{S}_L \tag{62}$$

$$\left(\frac{\partial F}{\partial\, {}^{\mathrm{RC}}_{a}D^{\alpha}_{b} x} \right)\Bigg|_{t=b} = 0, \qquad (n-1-j) \in \bar{S}_R. \tag{63}$$

In addition ${}^{R}_{a}D^{\alpha+j-n}_{b} x(a)$, $n-j \in S_L$ and ${}^{R}_{a}D^{\alpha+j-n}_{b} x(b)$, $n-j \in S_R$ must be specified. Note that once again the GBC in this case contains fractional derivative terms.

In the discussion so far, we considered all functions independent. In the next section, we consider the situation where the functions may be subjected to additional constraints.

6. The problem of Lagrange and the multiplier rule

In this section, we consider the problem of finding extremum of a functional defined in terms of several functions, not all of which are independent. When the functional and the constraints do not contain any fractional derivative term, the problem is known as the *problem of Lagrange*. In our case, functional and/or constraints will contain fractional derivative terms. For this reason, we will call such problems the *problems of Lagrange containing fractional derivatives*, or simply *fractional Lagrange problems*. The constraints among the functions can be of two types. When the constraints among the functions are described using some algebraic relations, we call them the *geometric constraints*, and when the relationships among the functions are described using FDEs, we call them the *dynamic constraints*. In this section, we first consider geometric constraints only and then dynamic constraints only. The case of mixed constraints is omitted. However, the formulation for mixed set of constraints could be obtained using the two cases presented below.

Case 1: Geometric constraints only

In this section, we consider the fractional Lagrange problem subjected to geometric constraints only. The problem can be defined as follows: find the extremum of the functional

$$\int_a^b F\left(t, x_1, \ldots, x_n, {}_a^{\mathrm{RC}}D_b^{\alpha} x_1, \ldots, {}_a^{\mathrm{RC}}D_b^{\alpha} x_n\right) \mathrm{d}t, \tag{64}$$

such that

$$\phi_i(t, x_1, \ldots, x_n) = 0, \qquad i = 1, \ldots, m < n \tag{65}$$

and the boundary conditions

$$x_{s1(j)}(a) = x_{s1(j)a}, \qquad n_{I1} \leqslant n - m \tag{66}$$

and

$$x_{s2(j)}(b) = x_{s2(j)b}, \qquad i = 1, \ldots, n_{I2} \leqslant n - m \tag{67}$$

where $s1$ and $s2$ are two sets containing n numbers from 1 to n which have been reordered, $sk(j)$ represents the jth element of set sk, $x_{s1(j)a}$ and $x_{s1(j)b}$, $j = 1, \ldots,$ are the terminal conditions, and n_{I1} and n_{I2} are the numbers of terminal conditions at the two ends, respectively. We assume that the constraint functions $\phi_i(t, x_1, \ldots, x_n) = 0, i = 1, \ldots, m$, are all independent. Note the following: (1) the order of elements in sets $s1$ and $s2$ is not important, except that the first n_{I1} numbers in $s1$ and the first n_{I2} numbers in $s2$ represent the numbers associated with the specified terminal conditions; (2) the numbers associated with the terminal conditions at the two ends need not be the same, and n_{I1} need not be equal to n_{I2}; (3) both n_{I1} and n_{I2} must be less than or equal to $n - m$. Otherwise either some of the terminal conditions would be redundant or they would be inconsistent with the constraints and (4) if either $n_{I1} < n - m$ or $n_{I2} < n - m$, the formulation leads to some GNBCs. For simplicity in the discussion to follow, we assume that $n_{I1} = n_{I2} = n - m$. Since $\phi_i(t, x_1, \ldots, x_n) = 0, i = 1, \ldots, m$, are independent, we can use equations (65)–(67) and a nonlinear solver such as the Newton–Raphson method to determine all functions at the end points. In the discussion to follow, we assume that all functions at the end points have been determined.

To develop the necessary conditions for the problem, assume that $x_j^*(t), j = 1, \ldots, n$, are the solutions to the above problem, and define

$$x_j(t) = x_j^*(t) + \epsilon\eta_j(t), \qquad j = 1, \ldots, n, \tag{68}$$

where ϵ is a sufficiently small number, and $\eta_j(t), j = 1, \ldots, n$, are functions consistent with the constraints, i.e. $x_j(t), j = 1, \ldots, n$, satisfy equation (65). Out of n functions $\eta_j(t)$, $n - m$ functions are independent. We assume that the first $n - m$ functions $\eta_j(t)$ are independent. Note that at the solution point, $\epsilon = 0$. From the above discussion, it follows that

$$\eta_j(a) = \eta_j(b) = 0, \qquad j = 1, \ldots, n. \tag{69}$$

Substituting equation (69) into equations (64) and (65) and differentiating them with respect to ϵ, setting ϵ and the resulting equations to 0, and using equations (20) and (69), we obtain

$$\int_a^b \sum_{j=1}^{n} \left[\frac{\partial F}{\partial x_j} - {}_a^{R}D_b^{\alpha} \frac{\partial F}{\partial {}_a^{\mathrm{RC}}D_b^{\alpha} x_j}\right] \eta_j \, \mathrm{d}t = 0 \tag{70}$$

and

$$\sum_{j=1}^{n} \frac{\partial \phi_i}{\partial x_j} \eta_j = 0, \qquad i = 1, \ldots, m. \tag{71}$$

Now, using the Lagrange multiplier technique, we obtain

$$\frac{\partial F}{\partial x_j} - {}_a^R D_b^\alpha \frac{\partial F}{\partial {}_a^{\mathrm{RC}} D_b^\alpha x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial \phi_i}{\partial x_j} = 0, \qquad j = 1, \ldots, n, \tag{72}$$

where λ_i, $i = 1, \ldots, m$, are the Lagrange multipliers.

Definition. *A set of admissible functions $x_j(t)$, $j = 1, \ldots, m$, are said to satisfy the multiplier rule if there exist multipliers $\lambda_i(t)$, $i = 1, \ldots, m$ on $[a, b]$ and a function*

$$\bar{F} = F + \sum_{i=1}^{m} \lambda_i \phi_i \tag{73}$$

such that

$$\frac{\partial \bar{F}}{\partial x_j} - {}_a^R D_b^\alpha \frac{\partial \bar{F}}{\partial {}_a^{\mathrm{RC}} D_b^\alpha x_j} = 0, \qquad j = 1, \ldots, n, \tag{74}$$

is satisfied along $x_j(t)$, $j = 1, \ldots, n$.

Theorem 5. *Every minimizing set of functions $x_j(t)$, $j = 1, \ldots, n$, must satisfy the multiplier rule.*

To prove this, substitute equation (73) into equation (74) and show that it leads to equation (72).

We now consider the dynamic constraints.

Case 2: Dynamic constraints only

In the case of dynamic constraints, equation (65) also contains fractional derivative terms. Here, we shall consider dynamic constraints which are linear in fractional derivative terms. Specifically, we will consider the dynamic constraints of the following form:

$$\phi_i\left(t, x_1, \ldots, x_n, {}_a^{\mathrm{RC}} D_b^\alpha x_k\right) = \bar{\phi}_i(t, x_1, \ldots, x_n) - {}_a^{\mathrm{RC}} D_b^\alpha x_k = 0, \qquad i = 1, \ldots, m < n, \tag{75}$$

where $k \in \{1, \ldots, n\}$. Further, one k occurs only once in equation (75). Such equations arise when the dynamics of a fractional system is given in a state-space form.

The multiplier rule discussed above is also applicable to the dynamic constraints considered here. This can be proved following the discussion given above and in [36, 37]. The necessary conditions for such type of problems defined using other type of fractional derivatives could be found in [26, 28, 29]. The necessary conditions for a system containing multiple fractional derivatives can be developed in a similar manner.

We now discuss the canonical form namely the Hamiltonian formulation of the Euler–Lagrange equations. Fractional Lagrangians and Hamiltonians have been recently considered in [21, 24]. However, they formulate the problems in terms of RLFDs and CFDs. In contrast, we present the formulation in terms of RCFDs. Once again, we omit the formulation in terms of RFDs as they can be derived in a similar manner.

7. The canonical form of the Euler–Lagrange equations

Let us now return to equation (49), which could be thought of as a system of $2n$ FDEs in terms of $2n$ functions $x_1, \ldots, x_n, {}_a^{\mathrm{RC}} D_b^\alpha x_1, \ldots, {}_a^{\mathrm{RC}} D_b^\alpha x_n$. To obtain a symmetrical and convenient form of these equations, define

$$p_j = \frac{\partial F}{\partial {}_a^{\mathrm{RC}} D_b^\alpha x_j}, \qquad j = 1, \ldots, n \tag{76}$$

and write ${}_a^{\mathrm{RC}}D_b^{\alpha}x_1, \ldots, {}_a^{\mathrm{RC}}D_b^{\alpha}x_n$ as functions of the variables $t, x_1, \ldots, x_n, p_1, \ldots, p_n$. This requires that the Jacobian

$$\frac{\partial p_1, \ldots, p_n}{\partial {}_a^{\mathrm{RC}}D_b^{\alpha}x_1, \cdots, {}_a^{\mathrm{RC}}D_b^{\alpha}x_n} = \det\left\| \frac{\partial^2 F}{\partial {}_a^{\mathrm{RC}}D_b^{\alpha}x_j \, \partial {}_a^{\mathrm{RC}}D_b^{\alpha}x_k} \right\| \tag{77}$$

be nonzero [35]. We assume that this condition is satisfied. In [35] it is stated that equation (77) guarantees only the local solvability of equation (76) with respect to ${}_a^{\mathrm{RC}}D_b^{\alpha}x_1, \ldots, {}_a^{\mathrm{RC}}D_b^{\alpha}x_n$, but it does not guarantee the possibility of representing ${}_a^{\mathrm{RC}}D_b^{\alpha}x_1, \ldots, {}_a^{\mathrm{RC}}D_b^{\alpha}x_n$ as functions of $t, x_1, \ldots, x_n, p_1, \ldots, p_n$, which are defined over the whole region under discussion. Therefore, the above considerations have local character. Since we have assumed that F and its partial derivatives with respect to its arguments are sufficiently smooth, we can assume that this representation is valid over the whole region under consideration.

We now express $F\big(t, x_1, \ldots, x_n, {}_a^{\mathrm{RC}}D_b^{\alpha}x_1, \ldots, {}_a^{\mathrm{RC}}D_b^{\alpha}x_n\big)$ in terms of a new function $H(t, x_1, \ldots, x_n, p_1, \ldots, p_n)$ related to F by the formula

$$H = -F + \sum_{j=1}^{n} p_j \, {}_a^{\mathrm{RC}}D_b^{\alpha}x_j. \tag{78}$$

When $\alpha = 1$, the function H is called the *Hamiltonian*, and the new variables $t, x_1, \ldots, x_n, p_1, \ldots, p_n, H$ are called the *canonical variables*. Because of their close similarity, we call H the *fractional Hamiltonian*, and the new variables $t, x_1, \ldots, x_n, p_1, \ldots, p_n, H$ the *fractional canonical variables*. In physics, variables $p_1, \ldots, p_n$ are also known as the *generalized momenta*.

We now show how the Euler–Lagrange equations (49) transform when we go over to fractional canonical variables. To accomplish this, we proceed as follows. By the definition of H, we have

$$\begin{aligned} \mathrm{d}H = {} & -\frac{\partial F}{\partial t}\,\mathrm{d}t - \sum_{j=1}^{n}\left(\frac{\partial F}{\partial x_j}\,\mathrm{d}x_j + \frac{\partial F}{\partial {}_a^{\mathrm{RC}}D_b^{\alpha}x_j} d\,{}_a^{\mathrm{RC}}D_b^{\alpha}x_j\right) \\ & + \sum_{j=1}^{n}\left(dp_j \, {}_a^{\mathrm{RC}}D_b^{\alpha}x_j + p_j \, d\,{}_a^{\mathrm{RC}}D_b^{\alpha}x_j\right). \end{aligned} \tag{79}$$

Using equation (76), this reduces to

$$\mathrm{d}H = -\frac{\partial F}{\partial t}\,\mathrm{d}t + \sum_{j=1}^{n}\left(-\frac{\partial F}{\partial x_j}\,\mathrm{d}x_j + \mathrm{d}p_j \, {}_a^{\mathrm{RC}}D_b^{\alpha}x_j\right). \tag{80}$$

This suggests that H is a function of $t, x_1, \ldots, x_n, p_1, \ldots, p_n$ only. Therefore, we can write

$$\mathrm{d}H = \frac{\partial H}{\partial t}\,\mathrm{d}t + \sum_{j=1}^{n}\left(\frac{\partial H}{\partial x_j}\,\mathrm{d}x_j + \frac{\partial H}{\partial p_j}\,\mathrm{d}p_j\right). \tag{81}$$

Comparing equations (80) and (81), and using equations (49) and (76), we obtain

$${}_a^{R}D_b^{\alpha}p_j = -\frac{\partial H}{\partial x_j}, \qquad {}_a^{\mathrm{RC}}D_b^{\alpha}x_j = \frac{\partial H}{\partial p_j}, \qquad j = 1, \ldots, n \tag{82}$$

and

$$\frac{\partial H}{\partial t} = -\frac{\partial F}{\partial t}. \tag{83}$$

Equations (82) represent $2n$ FDEs of order α for the system which is equivalent to the system (49). Because of their similarity with the canonical Euler equations for integer order systems, we call equations (82) the *fractional canonical system of Euler equations* or simply the *fractional canonical Euler equations*.

8. Variational problems involving multiple integrals

In the discussion so far, functions $x_j(t)$ were defined over one-dimensional space. In this section, we consider the problem of finding an extremum of a functional of functions defined over a multi-dimensional space. Such problems arise in field theories where the space and the time dimensions are considered simultaneously. For simplicity in the discussion to follow, we consider a functional which depends only on one function and its partial fractional derivatives. We assume that the function is defined over a two-dimensional rectangular domain, and the boundaries of the domain are fixed.

The variational problem involving multiple integrals can now be defined in the following way: *among all functions $x(t, z)$ which satisfy the boundary conditions*

$$\begin{cases} x(a_1, z) = x_{a1}(z), & x(b_1, z) = x_{b1}(z), \\ x(t, a_2) = x_{a2}(t), & x(t, b_2) = x_{b2}(t), \end{cases} \tag{84}$$

find the function for which the functional

$$J[x] = \int_{a2}^{b2} \int_{a1}^{b1} F\left(t, z, x, {}_{a1}^{\mathrm{RC}}D_{b1}^{\alpha} x, {}_{a2}^{\mathrm{RC}}D_{b2}^{\beta} x\right) \mathrm{d}t\, \mathrm{d}z \tag{85}$$

is an extremum, where $0 < \alpha, \beta < 1$. Here, ${}_{a1}^{\mathrm{RC}}D_{b1}^{\alpha} x$ and ${}_{a2}^{\mathrm{RC}}D_{b2}^{\beta} x$ are the Riesz–Caputo partial fractional derivatives of x of order α with respect to t and of order β with respect to z. The necessary conditions for this problem can be found using the approach presented in section 3. The result is:

Theorem 6. *Let $J[x]$ be a functional of the form given by equation (85) and defined on a set of functions $x(t, z)$ which have continuous Riesz–Caputo partial fractional derivatives of order α with respect to t and of order β with respect to z in $[a_1, b_1] \times [a_2, b_2]$ and which satisfy the boundary conditions given by equation (84). Then a necessary condition for $J[x]$ to have an extremum for a given function $x(t, z)$ is that $x(t, z)$ satisfy the following generalized Euler–Lagrange equation*

$$\frac{\partial F}{\partial x} - {}_{a1}^{R}D_{b1}^{\alpha} \frac{\partial F}{\partial {}_{a1}^{\mathrm{RC}}D_{b1}^{\alpha} x} - {}_{a2}^{R}D_{b2}^{\beta} \frac{\partial F}{\partial {}_{a2}^{\mathrm{RC}}D_{b2}^{\beta} x} = 0. \tag{86}$$

where ${}_{a1}^{R}D_{b1}^{\alpha}[*]$ and ${}_{a2}^{R}D_{b2}^{\beta}[*]$ are the Riesz partial fractional derivative of order α with respect to t and of order β with respect to z.

As an application of the above formulation, assume that F considered in equation (85) represents the Lagrangian density function, and define the generalized momenta as

$$p_\alpha = \frac{\partial F}{\partial {}_{a1}^{\mathrm{RC}}D_{b1}^{\alpha} x}, \qquad p_\beta = \frac{\partial F}{\partial {}_{a2}^{\mathrm{RC}}D_{b2}^{\beta} x}. \tag{87}$$

Following the approach presented in section 7, we can define the fractional Hamiltonian density function as

$$H = -F + p_\alpha\, {}_{a1}^{\mathrm{RC}}D_{b1}^{\alpha} x + p_\beta\, {}_{a2}^{\mathrm{RC}}D_{b2}^{\beta} x \tag{88}$$

Using the formulation presented in section 7, it can be demonstrated that for this case the fractional canonical Euler field equations (or the Hamilton field equations) are

$${}_{a1}^{\mathrm{RC}}D_{b1}^{\alpha} x = \frac{\partial H}{\partial p_\alpha}, \qquad {}_{a2}^{\mathrm{RC}}D_{b2}^{\beta} x = \frac{\partial H}{\partial p_\beta}, \tag{89}$$

and

$$ {}_{a1}^{R}D_{b1}^{\alpha} p_{\alpha} + {}_{a2}^{R}D_{b2}^{\beta} p_{\beta} = -\frac{\partial H}{\partial x}, \qquad \frac{\partial H}{\partial t} = -\frac{\partial F}{\partial t}. \tag{90} $$

Fractional field equations have been considered in [38], where formulations have been presented for the fractional Dirac field and the fractional Schrödinger equation. However, these formulations were done in terms of RLFDs.

Note that all theorems developed above for one-dimensional case can be extended for multi-dimensional cases. They are omitted since they follow the steps given in the previous sections.

9. Fractional optimal control formulation

As an additional application of the formulation presented in the previous sections, consider the following fractional optimal control of a time-invariant problem: find the control $u(t)$ which minimizes the quadratic performance index

$$ J(u) = \frac{1}{2}\int_0^1 [x^2(t) + u^2(t)]\,\mathrm{d}t \tag{91} $$

subjected to the system dynamics

$$ {}_{0}^{\mathrm{RC}}D_{1}^{\alpha} x(t) = -x(t) + u(t), \tag{92} $$

and the initial condition

$$ x(0) = 1. \tag{93} $$

For this example, we define (see equation (73))

$$ \bar{F} = \tfrac{1}{2}(x^2 + u^2) + \lambda\big(-x + u - {}_{0}^{\mathrm{RC}}D_{1}^{\alpha} x\big). \tag{94} $$

Using equation (74), we obtain the control equations as

$$ \begin{cases} {}_{0}^{\mathrm{RC}}D_{1}^{\alpha} x = -x + u, \\ \lambda + u = 0, \\ {}_{0}^{R}D_{1}^{\alpha} \lambda = -x + \lambda, \end{cases} \tag{95} $$

The above problem has been considered by the author in several papers where different formulations and different numerical schemes of the problem have been presented [19, 26, 28, 29]. The Euler–Lagrange equations developed here can be used to find fractional optimal control of time-varying system considered in these papers.

As an additional remark, it should be pointed out that finding closed-form solutions of the equations derived above is generally difficult, so a numerical technique may be necessary. This will be considered in the future.

10. Conclusions

Generalized Euler–Lagrange equations and the generalized natural boundary conditions have been presented for fractional variational problems defined in terms of Riesz and Riesz–Caputo fractional derivatives. Problems involving multiple functions, fractional derivatives of different orders, geometric and dynamic constraints, and multi-dimensional domains were considered. The formulations were applied to develop fractional momenta, fractional Hamiltonian, fractional Hamilton equations of motion, fractional field theory, and fractional optimal control. It is suggested that finding closed form solutions to these problems is difficult. Therefore, numerical techniques may be necessary to solve the resulting equations.

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